THE LESLIE MODEL
with
HARVESTING

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Introduction.

One of the simplest population growth models assumes that a population **P** is closed (no immigration or emmigration is allowed) and that each individual in **P** is identical with respect to survival rates and reproduction rates.

This model provides a good approximation when growth is gauged over a small interval of time and **P** is made up of single cell organisms which reproduce by dividing [11]. The model provides a poor approximation to a population's growth in general since populations consist of individuals with varying survival and reproduction rates. One model which was constructed in an attempt to remedy this problem is the Leslie matrix model. With the Leslie model, a population **P** is divided into age groups of equal time length called cohorts, where the reproduction and survival rates are allowed to vary between cohorts but not within a cohort. Furthermore, only the female portion of **P** is considered, although the same mathematical arguments would apply if both males and females were present and that ratio of males to females remained constant within each cohort [11]. This model, introduced by Lewis [10] and Leslie [9], is a discrete time model with a discrete age scale. The Leslie model has been applied to many different populations, one of which is the Hooded Seal of the North Atlantic Ocean. By using the Leslie model, Filpse and Veling were able to point out that hunting pressure in the years 1975 - 1979 was equal to or slightly greater than the Hooded Seal population could tolerate [5].

This paper covers several topics paramount to the proper maintenance of a closed population subject to harvesting. Section 0.0 introduces the *Leslie model*, states the *Perron - Frobenius Theorem* as it applies to the *Leslie matrix*, exposes some of the basic properties of the Leslie matrix, and then describes the long range behavior of a population when its corresponding Leslie matrix is *Primitive*. Section 1.0 develops the *Leslie model with harvesting,* and demonstrates some basic connections between the dynamics of a population whose growth is governed by a Leslie model without harvesting, and the dynamics of the same population with harvesting. These first two sections are subordinate to section 2.0 where all *reachable* and *holdable population states* are determined for a given Leslie matrix L and intial population . The climax of the paper occurs in section 3.0, where the idea of an *optimal harvesting scheme* is presented, and it is shown how an optimal harvesting scheme can be found with the methods of *linear programming.*

0.0 Mathematical formulation and basic results.

We partition **P** into m age classes $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ where $\mathbf{P}_i$, i=1,2, . . . , m consists of individuals in **P** of age i-1 to i time units. Each individual in $\mathbf{P}_i$ has a probability s_i of reaching $\mathbf{P}_{i+1}$ after 1 time unit and a birthrate of b_i. Now let $x_i(t)$ t=0,1,2, . . . represent the number of individuals in $\mathbf{P}_i$ at time t. Peilou has demonstrated that the dynamics of **P** can be captured by considering the first n age classes where $\mathbf{P}_n$ is the oldest age class with a nonzero birthrate [11]. Therefore we will assume m=n. The growth of **P** is then governed by following difference equations:

$$x_1(t+1) = \sum_{k=1}^{n} x_k(t) b_k$$

$$x_i(t) = x_{i-1}(t) s_{i-1} \qquad i = 2, 3, \ldots, n.$$

The above difference equations may be written as:

$$x(t+1) = Lx(t) \qquad t = 0,1,2,\ldots$$

where $x(\cdot) = [\, x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot)\,]^T \in M_{n,1}(\Re)$ and

$$L = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & s_{n-1} & 0 \end{bmatrix}.$$

We call $L \in M_n(\Re)$ a Leslie matrix, and $x(t)$ the population vector at time t. L is of course nonnegative with the following restrictions:

i) $b_i \geq 0, \qquad i = 1,2,\ldots,n-1 \qquad b_n > 0$

ii) $s_i > 0, \qquad i = 1,2,\ldots,n-1.$

We now present a list of some basic properties enjoyed by the Leslie matrix L [6].

0.1a **Property:** *Irreducibility.* The Leslie matrix is irreducible.

0.1b **Property.** *Characteristic polynomial.* The characteristic polynomial the Leslie matrix is given by

$$f(\lambda) = \lambda^n - b_1 \lambda^{n-1} - s_1 b_2 \lambda^{n-2} - \ldots - s_1 s_2 \ldots s_{n-2} b_{n-1} \lambda^1 - s_1 s_2 \ldots s_{n-1} b_n.$$

0.1c **Property:** *Nonsingularity.* The Leslie matrix is nonsingular.

0.1d **Property:** Let x , y ε $M_{n,1}(\Re)$ with $x \geq 0$, $y \geq 0$ and $x \geq y$. Then $Lx \geq Ly$.

Proof: Let x , y be as given. Then $x = y + y'$ for some $y' \varepsilon M_{n,1}(\Re)$, and thus $Lx = L(y + y') = Ly + Ly' \geq Ly$ since $Ly' \geq 0$. ◊

The most important result pertaining to the Leslie matrix is an application of the Perron - Frobenius theorem. The theorem applies to any nonegative irreducible matrix and thus it applies to the Leslie matrix [6], [7].

0.2 **Theorem:** Let $L \varepsilon M_n(\Re)$ be an irreducible Leslie matrix. Then

i) $\rho(L)$ is a simple eigenvalue of L.

ii) The eigenvector associated with $\rho(L)$ can be chosen to have positive components only.

$\rho(L)$ is refered to as the Perron root of L, and $\omega \varepsilon M_{n,1}(\Re)$, $\omega > 0$ such that $L\omega = \rho(L)\omega$ is called a Perron eigenvector of L.

In practice, it is important to know the behavior of x(t) as t tends to infinity. This long range behavior will of course depend on the matrix L. If L is a primitive matrix (L is irreducible and $\rho(L)$ is a strictly dominant eigenvalue of L) , then the following theorem applies [7].

0.3 **Theorem:** If $L \varepsilon M_n(\Re)$ is a primitive Leslie matrix, then

$\mathrm{Lim}_{t \to \infty} x(t) / \rho(L)^t = \omega z^T x(0)$, where $L\omega = \rho(L)\omega$, $L^T z = \rho(L)z$, and $\omega^T z = 1$.

Since the long range behavior of x(t) is easy to determine when L is primitive, it would be advantageous to have a theorem which characterizes primitive Leslie matrices. Theorem 0.4 provides such a characterization [11].

0.4 **Theorem:** Let $L \in M_n(\Re)$ be a Leslie matrix. Then L is primitive if and only if gcd (N) = 1 where $N = \{ k \,/\, b_k \neq 0,\ k = 1,2,\ldots,n \}$.

Thus in order to check the primitivity of L we need only calculate the greatest common divisor of the set of all subscripts corresponding to nonzero birthrates. It has been observed that populations whose growth is governed by nonprimitive Leslie matrices are rare [11].

1.0 The Leslie Model with Harvesting

In this section we investigate the population dynamics of a population **P** subject to harvesting, where the harvesting of **P** is carried out in the manner described by Beddington and Taylor [1]. We assume that in the absence of harvesting , the growth of **P** is governed by a leslie matrix $L \in M_n(\Re)$. Let x(0) be the initial population vector. A certain fraction of each age group in the initial population is to be harvested. Let $h_i(0)$ be the fraction of $x_i(0)$ harvested. Then the vector representing the remaining population is given by:

$$(I - D_{h(0)})\, x(0)$$

where $h(0) = [\, h_1(0), h_2(0), \ldots, h_n(0)\,]^T$ and $D_{h(0)} = \mathrm{diag}(\, h_1(0), h_2(0), \ldots, h_n(0)\,)$.

The growth of this remaining population is governed by the Leslie model so that

$$x(1) = L(I - D_{h(0)})\, x(0) .$$

We continue this harvesting procedure to obtain population vectors x(2), x(3), . . . where

$$x(t+1) = L(I - D_{h(t)})\, x(t) \qquad t = 0,1,2,\ldots ,$$

$h(t) \varepsilon M_{n,1}(\Re)$ is the age specific harvest vector at time t, and $D_{h(t)} = \text{diag}(h_1(t), h_2(t), \ldots, h_n(t))$. Naturally, there is the restriction

$$0 \le h_i(t) \le 1 \qquad i = 1,2,\ldots,n .$$

Here are some important results comparing L with $L(I - D_h)$, where $h \varepsilon M_{n,1}(\Re)$ and $0 \le h \le 1$. These results will be used extensively throughout the remainder this paper.

1.1 **Result :** $0 \le L(I - D_h) \le L$.

Proof: $0 \le L(I - D_h) = L - LD_h \le L$, since $LD_h \ge 0$. ◊

1.2 **Result :** Let $A,B \varepsilon M_n(\Re)$, $A \ge 0$ and $B \ge 0$ with $A \ge B$, then $\rho(A) \ge \rho(B)$.

A proof of Result 1.2 can be found in Horn and Johnson [7].

1.2 a **Result :** $\rho(L) \geq \rho(L(I - D_h))$. With equality if and only if $D_h = 0$.

Proof : By Result 1.1 and 1.2, $\rho(L) \geq \rho(L(I - D_h))$. Obviously if $D_h = 0$, then equality holds. Now assume that $\rho(L) = \rho(L(I - D_h))$. By Property 0.1b, the characteristic polynomials of L and $L(I - D_h)$ are $f(\lambda) = \lambda^n - b_1 \lambda^{n-1} - s_1 b_2 \lambda^{n-2} - \ldots - s_1 s_2 \ldots s_{n-2} b_{n-1} \lambda^1 - s_1 s_2 \ldots s_{n-1} b_n$ and $g(\lambda) = \lambda^n - (1-h_1)\, b_1 \lambda^{n-1} - (1-h_1)(1-h_2)\, s_1 b_2 \lambda^{n-2} - \ldots - (1-h_1)(1-h_2)\ldots(1-h_n)\, s_1 s_2 \ldots s_{n-2}\, s_{n-1} b_n$ respectively. Let $\alpha = \rho(L) = \rho(L(I - D_h))$. Then $f(\alpha) = g(\alpha) = 0$, so that $\alpha^n - f(\alpha) = \alpha^n - g(\alpha)$. Suppose that $h_1 > 0$. Then $b_1 \lambda^{n-1} \geq (1-h_1)\, b_1 \lambda^{n-1}, \ldots,$ $s_1 s_2 \ldots s_{n-2} b_{n-1} \lambda^1 \geq (1-h_1)(1-h_2)\ldots(1-h_{n-1})\, s_1 s_2 \ldots s_{n-2}\, b_{n-1}$, and $s_1 s_2 \ldots s_{n-1} b_n > (1-h_1)(1-h_2)\ldots(1-h_n)\, s_1 s_2 \ldots s_{n-1} \cdot b_n$ so that $\alpha^n - f(\alpha) < \alpha^n - g(\alpha)$, a contradiction. Similar contradictions arise when $h_i > 0$ for $i = 2, \ldots, n$. Hence $h = 0$. ◊

1.3 **Result :** $x(t) \leq L^t x(0)$ for all $t \in \mathbb{N}$.

Proof : The proof is by induction on t. Let $\mathbb{B} = \{ t \in \mathbb{N} \mid x(t) \leq L^t x(0) \}$. If $t = 0$, then $x(0) = L^0 x(0)$ so that $0 \in \mathbb{B}$. Now suppose that $k \in \mathbb{B}$. Then $x(k+1) = L(I - D_{h(k)})\, x(k) \leq L(I - D_{h(k)})\, L^k x(0) = L^{k+1} x(0) - L D_{h(k)} L^k x(0) \leq L^{k+1} x(0)$ since $L D_{h(k)} L^k x(0) \geq 0$. Therefore $k+1 \in \mathbb{B}$, and by the principle of mathematical induction, $x(t) \leq L^t x(0)$ for all $t \in \mathbb{N}$. ◊

The long range behavior of a population whose growth is governed by the Leslie model with harvesting is more difficult to determine than when there is no harvesting. The flexibility in the choice of harvest vectors h(0), h(1), h(2) . . . allows for many possibilities. However, in the case where $\rho(L) < 1$, the asymptotic behavior is easily determined [7].

1.4 **Theorem:** Let $A \varepsilon M_n(\Re)$. Then $\lim_{t\to\infty} A^t = 0$ if and only if $\rho(A) < 1$.

1.5 **Result :** If $\rho(L) < 1$, then $\lim_{t\to\infty} x(t) = 0$.

Proof: By result 1.3, $x(t) \leq L^t x(0)$ for all $t \varepsilon \mathcal{N}$, and by Theorem 1.4, $\lim_{t\to\infty} L^t x(0) = 0$, so that $\lim_{t\to\infty} x(t) = 0$. ◊

This result demonstrates that a population with $\rho(L) < 1$ is doomed. Most likely, a wildlife manager would avoid any harvesting of such a population and would initiate some management techniques to increase the birthrates or survival rates.

2.0 Reachability and Holdability

Wildlife managers are faced with the problem of controlling a population **P** to a desirable size and distribution. If **P** is so large that it is depleting the natural resources of its habitat, then some action must be taken to bring the size of **P** to a safe level. Suppose that $m \varepsilon M_{n,1}(\Re)$ is known to be a desirable population vector for **P**. We seek a harvesting scheme such that $x(k) = m$ for some $k \varepsilon \mathcal{P}$, where $x(0)$ is specified. If such a harvesting scheme exists, we say that m is *reachable*. A manager may further insist that $x(k+1) = m$ so that **P** reaches m and can be held at m. If such a harvesting scheme exists, we say that m is *holdable*. The aim of this section is to determine the set of reachable vectors and the set of holdable vectors for a given Leslie matrix L and initial population vector x(0).

In the upcoming definitions and results, L is assumed to be an irreducible Leslie matrix, x(t) represents the population vector at time t, h(t) is the vector of age specific harvest rates at time t, $D_{h(t)} = \mathrm{diag}(h_1(t), h_2(t), \ldots, h_n(t))$, $\rho(L)$ is the

Perron eigenvalue of L, and ω is a Perron eigenvector of L. Furthermore, assume that $x(0) \neq 0$, so that $L^t x(0) \neq 0$ for $t = 0, 1, 2, \ldots$.

2.1 **Definition :** A nonnegative vector $m \in M_{n,1}(\Re)$ is called *(L, x(0)) reachable in k steps* if there exists a finite sequence of vectors

$\mathcal{S} = \{ h(0), h(1), \ldots, h(k-1) \}$ $h(t) \in M_{n,1}(\Re)$, $0 \leq h(t) \leq 1$

$t = 0,1, \ldots, k-1$

such that $x(t+1) = L(I - D_{h(t)})\, x(t)$ and $x(k) = m$.

2.2 **Definition :** If $m \in M_{n,1}(\Re)$ is (L, x(0)) reachable in k steps for some $k \in \mathcal{P}$, then m is called *(L, x(0)) reachable.*

2.3 **Definition :** A nonnegative vector $m \in M_{n,1}(\Re)$ is said to be *(L, x(0)) holdable in k steps* if m is (L, x(0)) reachable in k steps and there is an $h(k) \in M_{n,1}(\Re)$ with $0 \leq h(k) \leq 1$, such that $x(k+1) = L(I - D_{h(k)})\, x(k) = x(k)$ and $x(k) = m$.

2.4 **Definition :** If $m \in M_{n,1}(\Re)$ is (L, x(0)) holdable in k steps for some $k \in \mathcal{P}$, then m is called *(L, x(0)) holdable.*

Armed with these definitions and the elementary results presented earlier, we are prepared to describe (L , x(0)) holdable sets and (L, x(0)) reachable sets. In the case where L is primitive, Theorem 2.8 gives a straightforword characterization of all (L, x(0)) reachable vectors where $\rho(L) > 1$ and Theorem 2.10 gives a characterization of all (L, x(0)) holdable vectors where $\rho(L) > 1$.

These two theorems are the most important since a wildlife manager would most likely not harvest a population whose corresponding Leslie matrix has $\rho(L) \leq 1$.

2.5 **Lemma :** If $m \in M_{n,1}(\Re)$ is $(L, x(0))$ reachable in k steps, then

$$\Sigma \leq m_1 \leq \Sigma + b_n e_n^T L^{k-1} x(0)$$

$$\text{where } \Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1}.$$

Proof: Assume that m is $(L, x(0))$ reachable in k steps, and let $x(k-1) = x$, $h = 1 - h(k-1)$. Then $m = L D_h x$ so that

$$m_1 = b_1 h_1 x_1 + b_2 h_2 x_2 + \ldots + b_n h_n x_n \qquad \text{equation 1}$$

and

$$m_2 = s_1 h_1 x_1$$

$$m_3 = s_2 h_2 x_2 \qquad \text{equations 2}$$

$$\ldots$$

$$m_n = s_{n-1} h_{n-1} x_{n-1}$$

Substituting equations 2 into equation 1 we obtain

$$m_1 = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1} + b_n h_n x_n .$$

By Result 1.3, $0 \leq b_n h_n x_n \leq b_n e_n^T L^{k-1} x(0)$, and the result follows. ◊

2.6 **Lemma :** Let $m \in M_{n,1}(\Re)$ $m \geq 0$, and let L be a primitive Leslie Matrix. If $\rho(L) > 1$, then there exists $J \in \mathcal{P}$ such that $L^j x(0) \geq m$ for all $j \in \mathcal{P}$ with $j \geq J$.

Proof : By Theorem 0.3, $\lim_{t\to\infty} L^t x(0) / \rho(L)^t = \omega z^T x(0)$ where ω , $z \in M_{n,1}(\Re)$ $\omega > 0$, $z > 0$, $L\omega = \rho(L)\omega$ and $L^T z = \rho(L)z$. Assume that $\rho(L) > 1$, then $\lim_{t\to\infty} L^t x(0) = \lim_{t\to\infty} \rho(L)^t \omega z^T x(0) = \infty$. Hence for any $m \in M_{n,1}(\Re)$ with $m > 0$, there exists a $J \in \mathcal{P}$ such that $L^j x(0) \geq m$ for all $j \in \mathcal{P}$ with $j \geq J$. ◊

2.7 **Theorem :** $m \in M_{n,1}(\Re)$ is $(L, x(0))$ reachable in j+1 steps if and only if the following two conditions hold:

i) $\Sigma \leq m_1 \leq \Sigma + b_n x_n$ where $x = L^j x(0)$, and $\Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1}$

ii) $m \leq L^{j+1} x(0)$.

Proof : Suppose that $m \in M_{n,1}(\Re)$ is reachable in j+1 steps. By Lemma 2.5 condition i) must hold. If condition ii) does not hold, then there exists an $i \in \{1,2,\ldots,n\}$ such that $e_i^T m > e_i^T L^{j+1} x(0)$. However, by Result 1.3, $x(j+1) \leq L^{j+1} x(0)$ which implies that $e_i^T x(j+1) \leq e_i^T L^{j+1} x(0) < e_i^T m$ so that m is not $(L, x(0))$ reachable in j+1 steps.

Now suppose that conditions i) and ii) hold and let $\mathcal{S} = \{ h(0), h(1), \ldots, h(j)\}$ where $h(0) = h(1) = \ldots = h(j-1) = 0$ and $h = 1 - h(j)$ is defined by

$$\begin{aligned} h_i &= 1 && \text{if } x_i = 0 \text{ and } i = 1,2,\ldots,n \\ &= m_{i+1}/s_i x_i && \text{if } x_i > 0 \text{ and } i = 1,2,\ldots,n-1 \\ &= (m_1 - \Sigma)/b_n x_n && \text{if } x_n > 0 \text{ and } i = n \end{aligned}$$

where $x = [x_1\ x_2 \ldots x_n]^T = x(j)$, and

$$\Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1}.$$

It is easily verified that the above choice for h(j) gives $m = L(I - D_{h(j)})\, x(j) = x(j+1)$. We next demonstrate that $0 \le h(j) \le 1$. It is clear by that by the choice of h, that $h(j) \le 1$, and if $x_i(j) = 0$, then $h_i(j) = 0$ for $i = 1, 2, \ldots, n$. If $x_i(j) > 0$ and $e_i^T h(j) < 0$, for $i \in \{1,2, \ldots, n-1\}$, then the preceding equation tells us that $e_{i+1}^T m = e_{i+1}^T (L - LD_{h(j)}) x(j) = e_{i+1}^T Lx(j) - h_i s_i x_i(j) \ge e_{i+1}^T Lx(j) = e_{i+1}^T L^j x(0)$, which is a contradition to condition ii. If $e_n^T h(j) < 0$ and $x_n(j) > 0$, then $(m_1 - \Sigma) > b_n x_n$, which contradicts condition i . Hence m is (L, x(0)) reachable in j+1 steps. ◊

Theorem 2.7 not only gives us sufficient and necessary conditions for $m \in M_{n,1}(\Re)$ to be a reachable vector, but the proof of the theorem has demonstrated a harvesting scheme which controls the population to m in j+1 steps (in the case where m is reachable in j+1 steps). The harvesting scheme consists of leaving the population unharvested at times 0 , 1, . . ., j-1 , then harvesting the population at time j so that $m = x(j+1)$.

2.8 **Theorem :** Assume that L is primitive. If $\rho(L) > 1$, then $m \in M_{n,1}(\Re)$ is reachable if and only if

$$\Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1} \le m_1 .$$

Proof : Assume that $\rho(L) > 1$, and let $\Sigma \le m_1$. Then by Lemma 2.6 there exists $J \in \mathbb{N}$ such that $m \le L^{j+1} x(0)$ for any $j \in \mathbb{N}$ with $j \ge J$. Also, by Lemma 2.6 there exists $J' \in \mathbb{N}$ such that $[\,0, 0, \ldots, (m_1 - \Sigma)/b_n]^T \le L^j x(0)$ for all $j \in \mathbb{N}$ with $j \ge J'$. Let J and J' be as given above and let $j \ge \max\{J,J'\}$. Then $m \le L^{j+1} x(0)$ and $e_n^T [\,0, 0, \ldots, (m_1 - \Sigma)/b_n]^T \le e_n^T L^j x(0)$. Consequently, by Theorem 2.7, m is

(L, x(0)) reachable. If $m \in M_{n,1}(\Re)$ is reachable then by condition i of Theorem 2.7, $\Sigma \le m_1$. ◊

2.9 **Lemma :** If $m \in M_{n,1}(\Re)$, $m > 0$ is (L, x(0)) reachable, then m is (L, x(0)) holdable if and only if the following two conditions hold :

i) $m \le Lm$

ii) $\Sigma \le m_1 \le \Sigma + b_n m_n$, where $\Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1}$.

Proof : Assume that m is (L, x(0)) reachable. We need only demonstrate that m is (L, m) reachable in 1 step if and only if the above two conditions hold. By Theorem 2.7, m is (L, m) reachable in 1 step if and only if $\Sigma \le m_1 \le \Sigma + b_n e_n^T m$ and $m \le Lm$. These are precisely the two conditions found above. ◊

2.10 **Theorem :** Assume that L is primitive, and let $\rho(L) > 1$. Then $m \in M_{n,1}(\Re)$, $m > 0$ is (L, x(0)) holdable if and only if conditions i and ii of Lemma 2.9 hold.

Proof : If m is (L, x(0)) holdable then by Lemma 2.9, conditions i and ii must hold. Assume now that $\rho(L) > 1$ and conditions i and ii hold. Then by Theorem 2.8, m is (L, x(0)) reachable, and thus by Lemma 2.9, m is (L, x(0)) holdable. ◊

We next turn our attention to the less important case where $\rho(L) \le 1$.

2.11 **Theorem :** If $\rho(L) < 1$, then 0 is the only (L, x(0)) holdable population vector.

Proof : Let $\rho(L) < 1$ and let $m \ \varepsilon \ M_{n,1}(\Re)$ be a holdable vector. Then there exists $h \ \varepsilon \ M_{n,1}(\Re)$, $0 \leq h \leq 1$ such that $m = L(I - D_h)m$. This implies that either $m = 0$, or m is an eigenvector of $L(I - D_h)$ with corresponding eigenvalue of 1. However by Result 1.2, $1 > \rho(L) \geq \rho(L(I - D_h))$ which makes an eigenvalue of 1 impossible for $L(I - D_h)$. Hence $m = 0$ is the only possible holdable population vector. If we let $h(0) = 1$, then we see that $m = 0$ is holdable in 1 step. ◊

2.12 **Theorem :** If $\rho(L) = 1$ and $m \ \varepsilon \ M_{n,1}(\Re)$ is (L, x(0)) holdable, then $m = c\omega$ where $c \ \varepsilon \ \Re$, and $c \geq 0$.

Proof : Suppose $\rho(L) = 1$ and that m is (L , x(0)) holdable. Then there exists $h \ \varepsilon \ M_{n,1}(\Re)$, $0 \leq h \leq 0$ such that $m = L(I - D_h)m$. This implies that either $m = 0$ or m is an eigenvector of $L(I - D_h)$ with corresponding eigenvalue 1. If $m = 0$, then $m = 0\ \omega$. If $m \neq 0$, then by result 1.2a, $1 = \rho(L) \geq \rho(L(I - D_h)) = 1$ which implies that $D_h = 0$. Hence m is a Perron eigenvector of L so that $m = c\omega$ for some $c \ \varepsilon \ \Re$ with $c \geq 0$. ◊

A population **P** is said to be *stable* if the percentage of the population in each age group is constant over time. In the case that **P**'s growth is governed by a Leslie model without harvesting, the population is stable at time t if and only if $x(t+1) = Lx(t) = cx(t)$ for some $c \ \varepsilon \ \mathcal{R}$, $c \geq 0$. If such a c exists and $x(t) \geq 0$, then it can be shown that $c = \rho(L)$ and $Lx(t) = \rho(L)x(t)$ [6]. Notice that the growth of **P** in a stable state is completely determined by $\rho(L)$. When $\rho(L) \geq 1$ and **P** is stable, then **P** is increasing in each of its age groups. When $\rho(L) \leq 1$ and **P** is stable, then **P** is decreasing in each of its age groups. We next characterize all vectors $x \ \varepsilon \ M_{n,1}(\Re)$, $x > 0$ such that $Lx \geq x$, and all vectors $y \ \varepsilon \ M_{n,1}(\Re)$, $y > 0$ such that

$Ly \leq y$.

2.13 **Definition :** Let $x \,\varepsilon\, M_{n,1}(\Re)$, $x \geq 0$, $x \neq 0$. x is called an *L - increasing* vector if $Lx \geq x$. If $Lx \leq x$, then x is called *L - decreasing*. The set of all L - increasing vectors is denoted by $\mathcal{I}(L)$, and the set of all L - decreasing vectors is denoted by $\mathcal{D}(L)$.

2.14 **Result :** Let L have Perron root $\rho(L)$ and Perron eigenvector ω. If $\rho(L) \leq 1$, then $\omega \,\varepsilon\, \mathcal{D}(L)$. If $\rho(L) \geq 1$, then $\omega \,\varepsilon\, \mathcal{I}(L)$.

2.15 **Theorem :** Let $x \,\varepsilon\, M_{n,1}(\Re)$, $x \geq 0$. (i) If $Lx > \alpha x$ for some $\alpha \,\varepsilon\, \Re$ with $\alpha > 0$, then $\rho(L) > \alpha$. Also, (ii) if $Lx < \alpha x$, then $\rho(L) < \alpha$.

Proof : Assume that $x \,\varepsilon\, M_{n,1}(\Re)$, $x \geq 0$ and $Lx > \alpha x$ for some $\alpha \,\varepsilon\, \Re$. Then there exists $z \,\varepsilon\, M_{n,1}(\Re)$, $z > 1$ such that $Lx = D_z \alpha Lx$. This implies that $\alpha = \rho(D_z^{-1}L)$. Now by similarity, $\rho(D_z^{-1}L) = \rho(D_z D_z^{-1}L D_z^{-1}) = \rho(L D_z^{-1})$. Notice that $L D_z^{-1} \leq L$ so that by Result 1.2a, $\rho(L D_z^{-1}) \leq \rho(L)$ with equality only if $D_z^{-1} = I$. Note that $D_z^{-1} < I$, so that $\rho(L D_z^{-1}) < \rho(L)$. Therefore $\rho(L) > \alpha$.

Next assume that x is as given and $Lx < \alpha x$ for some $\alpha \,\varepsilon\, \Re$. Then there exists $z \,\varepsilon\, M_{n,1}(\Re)$, $z > 1$ such that $\alpha x = D_z Lx$. This implies that $\alpha = \rho(D_z L)$. By similarity $\rho(D_z L) = \rho(D_z^{-1} D_z L D_z) = \rho(L D_z)$. By Result 1.2 $\rho(L D_z) \geq \rho(L)$, and using an argument similar to that used Result 1.2a, equality holds if and only if $D_z = I$. Note that $D_z > I$ so that $\rho(L) < \alpha$. ◊

2.16 **Corollary :** (i) If $\rho(L) > 1$, then $\mathfrak{D}(L) = \phi$. (ii) If $\rho(L) < 1$, then $\mathfrak{I}(L) = \phi$.

(iii) If $\rho(L) = 1$, then $\mathfrak{D}(L) = \mathfrak{I}(L) = \{ x \in M_{n,1}(\Re) | \ x = c\omega \ , c \in \Re \quad c > 0 \}$.

Proof : Let $x \in M_{n,1}(\Re)$, $x \geq 0$. If $x \in \mathfrak{D}(L)$ and $Lx < x$, then by theorem 2.15 (i), $\rho(L) < 1$. If $x \in \mathfrak{D}(L)$ and $Lx = x$, then $\rho(L) = 1$. Thus by the contrapositive, (i) holds. Similarly, (ii) follows from theorem 2.15 (ii) and the fact that if $Lx = x$, then $\rho(L) = 1$.

Finally, if $\rho(L) = 1$ and $Lx = x$, then by the Perron - Frobenius theorem, there exists $\omega > 0$ such that $x = c\omega$ so that $c\omega \in \mathfrak{D}(L)$ and $c\omega \in \mathfrak{I}(L)$ for some $c \in \Re$, $c > 0$. Theorem 2.15 disallows the possibilities (a) $\rho(L) = 1$ and $Lx > x$, and (b) $\rho(L) = 1$ and $Lx < x$. Therefore $\mathfrak{D}(L) = \mathfrak{I}(L) = \{ x \in M_{n,1}(\Re) \mid x = c\omega, c \in \Re$ $c > 0 \}$. ◊

2.17 **Theorem :** Let $x \in M_{n,1}(\Re)$, $x \geq 0$, $x \neq 0$. Then $x \in \mathfrak{I}(L)$ if and only if there exists $\beta \in M_{n,1}(\Re)$, $0 < \beta \leq 1$ such that $D_\beta Lx = x$, and $\beta_i = 1$ whenever $x_i = 0$.

Proof : First assume that $x \in \mathfrak{I}(L)$ so that $Lx \geq x$. This gives rise to the following inequalities:

$$\sum_{i=1}^{n} b_i x_i \geq x_1 \quad \text{and} \quad s_i x_i \geq x_{i+1} \qquad i = 1,2,3,\ldots, n-1.$$

Therefore, there exists $c \in M_{n,1}(\Re)$, $c \geq 1$ such that

$$\sum_{i=1}^{n} b_i x_i = c_1 x_1 \quad \text{and} \quad s_i x_i = c_{i+1} x_{i+1} \quad i = 1,2,\ldots, n-1.$$

Let $c_i = 1$ whenever $x_i = 0$.These equations can be written as $Lx = D_c x$. Let $D_\beta = D_c^{-1}$ so that $D_\beta Lx = x$,$0 < \beta \leq 1$ and $\beta_i = 1$ whenever $x_i = 0$.

Now suppose that $\beta \varepsilon M_{n,1}(\Re)$, $0 < \beta \leq 1$ and $D_\beta Lx = x$. Then $Lx = D_\beta^{-1}x$, where $D_\beta^{-1} \geq I$ so that $D_\beta^{-1}x \geq x$ which implies that $Lx \geq x$. ◊

The next theorem is a sister theorem to Theorem 2.17. Its proof is constructed using the same methods used in Theorem 2.17.

2.18 **Theorem :** Let $x \varepsilon M_{n,1}(\Re)$, $x \geq 0$, $x \neq 0$. Then $x \varepsilon \mathfrak{D}(L)$ if and only if there exists $\beta \varepsilon M_{n,1}(\Re)$ with $\beta \geq 1$ such that $D_\beta Lx = x$, and $\beta_i = 1$ whenever $x_i = 0$.

2.19 **Lemma :** If $x \varepsilon M_{n,1}(\Re)$, $x \geq 0$, $x \neq 0$ is such that $x \varepsilon \mathcal{I}(L)$, then $L^t x \varepsilon \mathcal{I}(L)$ for all $t \varepsilon \mathcal{N}$.

Proof : The argument is made by induction. Let $\mathcal{V} = \{ t \varepsilon \mathcal{N} \mid L^t x \varepsilon \mathcal{I}(L)$ where $x \varepsilon \mathcal{I}(L) \}$. $0 \varepsilon \mathcal{V}$ since it is assumed that $x \varepsilon \mathcal{I}(L)$. Suppose now that $k \varepsilon \mathcal{I}(L)$.By Theorem 2.17, there exists $\beta \varepsilon M_{n,1}(\Re)$, $0 < \beta \leq 1$ such that $D_\beta L(L^k x) = L^k x$. Therefore $L(L^{k+1}x) = L(D_\beta^{-1}L^k x) = L((D_\beta^{-1} - I + I)L^k x) = L((D_\beta^{-1} - I)L^k x + L^k x) = L(D_\beta^{-1} - I)L^k x + L^{k+1}x \geq L^{k+1}x$ since $L(D_\beta^{-1} - I)L^k x \geq 0$. Hence $k+1 \varepsilon \mathcal{V}$ and by mathematical induction, $L^t x \varepsilon \mathcal{I}(L)$ for all $t \varepsilon \mathcal{N}$. ◊

2.20 **Lemma :** If $x \varepsilon M_{n,1}(\Re)$, $x \geq 0$, $x \neq 0$ is such that $x \varepsilon \mathfrak{D}(L)$, then $L^t x \varepsilon \mathfrak{D}(L)$ for all $t \varepsilon \mathcal{N}$.

The proof of Lemma 2.20 is analogous to the proof of Lemma 2.19 and is omitted. The two results tell us that in the absence of harvesting, once a population begins to increase (decrease) over time in each age group, it continues to do so.

Often times it is important to know the minimum amount of time that it takes for a population subject to harvesting to be controlled to a specified population vector m. This is especially true when the cost of having the population away from a desirable population is high.

2.21 **Definition :** Let $m \varepsilon M_{n,1}(\Re)$ be (L, x(0)) reachable. $\chi(L, x(0), m) = \min\{ k \varepsilon \mathcal{P} \mid m$ is (L , x(0)) reachable in k steps $\}$.

2.22 **Theorem :** Assume that $x(0) \varepsilon \mathcal{I}(L)$, $x(0) > 0$ and $m \varepsilon M_{n,1}(\Re)$ is (L, x(0)) reachable, then m is (L , x(0)) reachable in k steps for any $k \varepsilon \mathcal{P}$ with $k \geq \chi(L, x(0), m)$.

Proof : Let $m \varepsilon M_{n,1}(\Re)$ be (L, x(0)) reachable. Then by Theorem 2.7,

$$\Sigma \leq m_1 \leq b_n e_n^T L^{\chi-1} x(0) + \Sigma$$

$$\text{where } \Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1} \quad \text{and } \chi = \chi(L,x(0),m).$$

Since $x(0) \varepsilon \mathcal{I}(L)$, Theorem 2.19 tells us that $L^t x(0) \varepsilon \mathcal{I}(L)$ for all $t \varepsilon \mathcal{N}$. Therefore, $m \leq L^{\chi} x(0) \leq L^{\chi+1} x(0) \leq L^{\chi+2} x(0) \leq \ldots$, and $m_1 \leq b_n e_n^T L^{\chi-1} x(0) + \Sigma \leq b_n e_n^T L^{\chi} x(0) + \Sigma \leq \ldots$. Thus by Theorem 2.7, m is (L, x(0)) reachable in k steps for each $k \varepsilon \mathcal{P}$ with $k \geq \chi$. ◊

2.23 **Theorem :** Assume that $x(0) \varepsilon \mathcal{D}(L)$, $x(0) > 0$ and that $m \varepsilon M_{n,1}(\Re)$ is (L, x(0)) reachable in k steps. Then m is (L, x(0)) reachable in j +1 steps, where $j \varepsilon \{ 0, 1, \ldots , k-2\}$.

Proof : Since m is assumed to be (L, x(0)) reachable in k steps, by Theorem 2.7,

$$\text{i) } \Sigma + b_n e_n^T L^{k-1} x(0) \geq m_1 \geq \Sigma \quad \text{and ii) } m \leq L^k x(0) .$$

$$\text{where } \Sigma = \sum_{i=1}^{n-1} \frac{b_i}{s_i} m_{i+1} .$$

Furthermore , by Lemma 2.20, $m \leq L^k x(0) \leq L^{k-1} x(0) \leq \ldots \leq L^1 x(0) \leq x(0)$, and $m_1 \leq b_n e_n^T L^{k-1} x(0) + \Sigma \leq b_n e_n^T L^{k-2} x(0) + \Sigma \leq \ldots \leq b_n e_n^T L^0 x(0) + \Sigma$. Therefore conditions i and ii of Theorem 2.7 are satisfied for j = 0,1, . . ., k-2. ◊

It follows from Theorem 2.23 that $\chi(L,x(0),m) = 1$ in the case that $x(0) \varepsilon \mathcal{D}(L)$ and m is (L, x(0)) reachable. In general, it is more difficult to determine $\chi(L,x(0),m)$. One way to determine $\chi(L, x(0), m)$ is by use of Theorem 2.7. If m is (L, x(0)) reachable, then by Theorem 2.7, $\chi(L,x(0),m) = \min\{ k \,\varepsilon\, \mathcal{P} \mid L^k x(0) \geq m, \text{ and } (m_1 - \Sigma)/b_n \leq e_n^T L^{k-1} x(0) \}$. This indicates that $\chi(L,x(0),m)$ can be determined by evaluating $L^k x(0)$ at k = 1,2,3, . . . , and finding the smallest such k such that $L^k x(0) \geq m$. It would be nice to have a method for determining $\chi(L,x(0),m)$ that does not require raising L to powers. The following is a special case where $\chi(L,x(0),m)$ is easily determined without raising L to powers.

2.24 **Result :** Assume that m is (L,x(0)) reachable. If $x(0) = \omega$, and $m = c\omega$ for some $c \,\varepsilon\, \Re$, $c \geq 0$, then $\chi(L,x(0),m) = \min\{ k \,\varepsilon\, \mathcal{P} \mid \rho(L)^k \geq c \text{ and } (m_1 - \Sigma)/b_n \leq \rho(L)^{k-1} \omega_n \}$.

Proof : By Theorem 2.7, $\chi(L,x(0),m) = \min\{ k \,\varepsilon\, \mathcal{P} \mid L^k x(0) \geq m$, and

$(m_1 - \Sigma)/b_n \leq e_n^T L^{k-1} x(0) \}$. By the above choice of m and x(0), $L^k x(0) = L^k \omega = \rho(L)^k \omega$. Consequently,

$$L^k x(0) \geq m$$

$$\Leftrightarrow \quad \rho(L)^k \omega \geq m = c\omega$$

$$\Leftrightarrow \quad \rho(L)^k \geq c, \text{ and}$$

$$(m_1 - \Sigma)/b_n \leq e_n^T L^{k-1} x(0)$$

$$\Leftrightarrow \quad (m_1 - \Sigma)/b_n \leq e_n^T \rho(L)^{k-1} \omega$$

Therefore, $\{ k \in \mathcal{P} \mid L^k x(0) \geq m \text{ and } (m_1 - \Sigma)/b_n \leq e_n^T L^{k-1} x(0) \} = \{ k \in \mathcal{P} \mid \rho(L)^k \geq c$ and $(m_1 - \Sigma)/b_n \leq \rho(L)^{k-1} \omega_n \}$, and the result follows. ◊

Result 2.24 can be generalized to obtain an upper bound for $\chi(L,x(0),m)$. Theorem 2.25 provides the generalization.

2.25 **Theorem :** Assume that $m \in M_{n,1}(\Re)$ is (L,x(0) reachable and let $c = \max \{ r \in \Re \mid r\omega \leq x(0) \}$. Then $\chi(L,x(0),m) \leq \min\{ k \in \mathcal{P} \mid c\rho(L)^k \omega \geq m$ and $(m_1 - \Sigma)/b_n \leq c\rho(L)^{k-1} \omega_n \}$.

Proof : Since $c\omega \leq x(0)$, by property 0.1d, $L^k x(0) \geq L^k c\omega = c\rho(L)^k \omega$. Therefore, if $c\rho(L)^k \omega \geq m$, then $L^k x(0) \geq m$, and if $(m_1 - \Sigma)/b_n \leq c\rho(L)^{k-1} \omega_n$, then $(m_1 - \Sigma)/b_n \leq L^{k-1} x(0)$ so that $\mathcal{V}_2 = \{ k \in \mathcal{P} \mid L^k x(0) \geq m \text{ and } (m_1 - \Sigma)/b_n \leq L^{k-1} x(0) \} \supseteq \mathcal{V}_1 = \{ k \in \mathcal{P} \mid c\rho(L)^k \omega \geq m \text{ and } (m_1 - \Sigma)/b_n \leq c\rho(L)^{k-1} \omega_n \}$. Hence $\chi(L,x(0),m) = \min \mathcal{V}_2 \leq \min \mathcal{V}_1$. ◊

3.0 Optimal Harvesting Strategies

In section 2.0 we characterized the set of population vectors which are (L,x(0)) reachable for a given Leslie matrix L and initial population vector x(0). If m is (L,x(0)) reachable in k steps, then there exists a nonempty set $\mathcal{A}_k(m) = \{ \mathcal{S} \mid \mathcal{S}$ is a finite sequence of harvest vectors $\{ h(0), h(1), \ldots, h(k-1) \}$ such that m is reachable through $\mathcal{S} \}$. Since m is assumed reachable, $\mathcal{A}_k(m)$ is guaranteed to be nonempty. Assume that each individual in the ith age class of the harvested population has a value of v_i when it is harvested. We then defined the vector $v \in M_{n,1}(\mathfrak{R})$ by $v = [v_1, v_2, \ldots, v_n]^T$, which is known as the value vector and is assumed to be constant over time. The harvest of the initial population has a value of $v^T D_{h(0)} x(0)$, and the harvest of the population at time j has a value of $v^T D_{h(j)} x(j)$, where $j \in \mathcal{P}$. From time 0 to time k-1 then, there is a total value of

$$\sum_{i=0}^{k-1} v^T D_{h(i)} x(i)$$

which we intend to maximize over the set $\mathcal{A}_k(m)$; this quantity is known as the k step yield.

3.1 **Definition :** For a given finite sequence of harvest vectors $\mathcal{S} = \{ h(s), h(s+1), \ldots, h(k-1) \}$,

$$\sum_{i=s}^{k-1} v^T D_{h(i)} x(i) = y(s,k-1)$$

is called the *yield over the time interval [s,k-1]*.

The problem we have described above is known as an *optimal contol* problem with *fixed endpoints* [2]. The fixed endpoints are x(0) and m = x(k), where m is assumed to be (L,x(0)) reachable in k steps. In the language of *control theory*, { x(0), x(1), x(2), . . . } is known as the *trajectory*, the set $\mathcal{A}_k(m)$ is the set of *allowable controls* , and y(0, k-1) is the *objective* which we intend to maximize [2]. The problem stated in language of optimal control is as follows : *For a given Leslie matrix L , initial population vector x(0), and (L,x(0)) reachable vector m, find controls h(0), h(1), . . ., h(k) such that x(k) = m, and y(0,k-1) is as large as possible.* Furthermore, we say that $\mathcal{S}$ = { h(s), h(s+1), . . . , h(k-1) } is an *optimal x(s) - m (k-s)-step harvesting policy* if m =x(k) and y(s,k-1) is at its maximum, we will denote the set of all such harvesting policies by $\mathcal{A}_{s,k}(m)$.

Many authors have analyzed the problem of maximizing the *sustainable yield* of a population when $\rho(L) \geq 1$ [1], [4], [8], [12], [13]. In this problem, it is insisted that the population is harvested in such a way that the population distribution and size is constant. The problem consists of maximizing $v^T x$ over all vectors $x \in M_{n,1}(\Re)$ such that x is (L,x) holdable in 1 step and $1^T x = 1$.

The *Maximum Sustainable Yield* problem can easily be posed as a linear program [1]. The optimal control problem with fixed endpoints described above can also be solved using linear programming. The following theorem is used to set up the constraints involved in the linear program.

3.2 **Theorem :** Let $m \in M_{n,1}(\Re)$ be (L,x(0)) reachable in k steps, and let $\mathcal{S}$ = { h(0), h(1), . . . , h(k-1) } with $h(t) \in M_{n,1}(\Re)$, $x(t+1) = L(I - D_{h(t)})x(t)$, t = 0, 1, . . . , k-1, where $h_i(t) = 0$ when $x_i(t) = 0$. Then $\mathcal{S} \in \mathcal{A}_k(m)$ if and only if

$$\text{i) } x(k) = m$$

and $$\text{ii) } 0 \le L^{-1}x(t) \le x(t\text{-}1) \qquad t = 1, 2, \ldots, k.$$

Proof : Assume that $\mathcal{S} = \{ h(0), h(1), \ldots, h(k\text{-}1) \} \in \mathcal{A}_k(m)$. Then $m = x(k)$ follows immediately, and for $t = 0, 1, \ldots, k$, $L^{-1}x(t) = (I - D_{h(t-1)})x(t\text{-}1) =$ $x(t\text{-}1) - D_{h(t-1)}x(t\text{-}1) \le x(t\text{-}1)$ since $D_{h(t-1)}x(t\text{-}1) \ge 0$, and $L^{-1}x(t) \ge 0$ since $D_{h(t-1)} \le 1$. Thus condition i) holds.

Now assume that conditions i) and ii) hold for $\mathcal{S}$. We need only show that $0 \le h(t) \le 1$. By condition i),

$$0 \le L^{-1}x(t) \le x(t\text{-}1)$$

$$\Rightarrow \qquad 0 \le L^{-1}(L\,(I - D_{h(t-1)})x(t\text{-}1)) \le x(t\text{-}1)$$

$$\Rightarrow \qquad 0 \le (I - D_{h(t-1)})x(t\text{-}1)) \le x(t\text{-}1).$$

Therefore, $0 \le e_i^T(I - D_{h(t-1)})x(t\text{-}1) = (1 - h_i(t\text{-}1))x_i(t\text{-}1) \le x_i(t\text{-}1)$ for $i = 1, 2, \ldots, n$. If $x_i(t\text{-}1) > 0$, then $0 \le (1 - h_i(t\text{-}1)) \le 1$, so that $0 \le h_i(t\text{-}1) \le 1$. If $x_i(t\text{-}1) = 0$, then by choice of $\mathcal{S}$, $h_i(t\text{-}1) = 0$. Hence $\mathcal{S} = \{ h(0), h(1), \ldots, h(k\text{-}1) \} \in \mathcal{A}_k(m)$. ◊

One more result is needed to present the problem as a linear program. It is a result that allows us to write the yield as a function of the trajectory.

3.3 **Result :** Let L, $x(0)$, and v^T be given. If $\mathcal{S} = \{h(0), h(1), \ldots, h(k\text{-}1)\}$ is a k step harvesting policy with corresponding trajectory $\mathcal{T} = \{x(0), x(1), \ldots, x(k)\}$, then

$$y(0,k-1) = v^T x(0) - v^T x(k) + \sum_{t=1}^{k-1} v^T (I - L^{-1})\, x(t)\ .$$

Proof : By definition, $y(0,k-1) = v^T(D_{h(0)}x(0) + D_{h(1)}x(1) + \ldots + D_{h(k-1)}x(k-1))$. Now $x(t+1) = L(I - D_{h(t)})x(t)$, so that $D_{h(t)}x(t) = x(t) - L^{-1}x(t+1)$ for $t = 0,1, \ldots ,k-1$. Substituting these equations into the above expression for $y(0,k-1)$, we have $y(0,k-1) = v^T(x(0) - L^{-1}x(1) + x(1) - L^{-1}x(2) + \ldots + x(k-1) - L^{-1}x(k))$ so that $y(0,k-1) = v^Tx(0) - v^Tx(k) + v^T((I - L^{-1})x(1) + (I - L^{-1})x(2) + \ldots + (I - L^{-1})x(k-1))$. ◊

In the optimization problem of interest , x(0) and x(k) are assumed fixed so that by the above result, y(0,k-1) is at a maximum if and only if

$$v^T \sum_{t=1}^{k-1} (I - L^{-1})x(t) \quad \text{is at its maximum.}$$

3.4 ***THE LINEAR PROGRAM***

Let x(0) and L be given, and let m be (L,x(0)) reachable in k steps. We are interested in maximizing y(0,k-1) with respect to the set of all possible trajectories $\mathfrak{B} = \{ \mathcal{T} \mid$ There exists $\mathcal{S} \in \mathcal{A}_k(m)$ which has corresponding trajectory $\mathcal{T} = \{ x(0), x(1), \ldots , x(k) \} \}$. By Theorem 3.2, $\mathcal{T} \in \mathfrak{B}$ if and only if the two conditions of Theorem 3.2 are satisfied. Since m is assumed to be (L,x(0)) reachable in k steps, $\mathfrak{B}$ is certainly nonempty. Also, since $\mathfrak{B}$ is closed and bounded and y(0,k-1) is continuous over $\mathfrak{B}$, y(0,k-1) takes on a maximum value over $\mathfrak{B}$ [2]. Since the constraints on $\mathcal{T} \in \mathfrak{B}$ are linear in $\mathcal{T}$, and y(0,k-1) is linear in $\mathcal{T}$, the methods of linear programming may be applied to find an optimal $\mathcal{T}$ say $\mathcal{T}'$. Once a $\mathcal{T}' = \{x(0)', x(1)', \ldots, x(k-1)', x(k)'\}$ is discovered, the equations

$$x(t+1)' = L(I - D_{h(t)'})x(t)' \qquad t = 0, 1, \ldots, k-1$$

may be used to solve for $\mathcal{S}' = \{ h(0)', h(1)', \ldots, h(k)' \}$, letting $h_i(t) = 0$ whenever $x_i(t) = 0 \;\; i = 1,2, \ldots, n$.

The following is a formal statement of the problem in the language of linear programming [3].

$$\text{maximize} : \sum_{t=1}^{k-1} v^T(I - L^{-1})x(t) \; , \quad \text{where } x(t+1) = L(I - D_{h(t)})x(t)$$

$$\text{subject to} : \quad \text{i)} \quad x(k) = m$$

$$\text{ii)} \quad 0 \leq L^{-1}x(t) \leq x(t-1) \qquad t=1,2,\ldots,k$$

Example 3.5 will demonstrate the use of linear programming and Theorem 2.7 to solve an optimal yield problem.

3.5 **Example :**

$$\text{Let } L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; x(0) = \begin{bmatrix} 20 \\ 8 \end{bmatrix}, \text{and } m = \begin{bmatrix} 10 \\ 5 \end{bmatrix}.$$

a) Determine whether or not m is (L,x(0)) reachable in 2 steps. Is m also (L,x(0)) holdable in 2 steps?

b) If m is (L,x(0)) reachable in 2 steps, determine an optimal 2 - step harvesting strategy.

Solution : a) We use Theorem 2.7. Notice that $\Sigma = b_1 m_2 / s_1 = 0$ and $e_2^T L x(0) = e_2^T [\, 8 \;\; 20 \,]^T = 20$, so that $\Sigma \leq m_2 = 5 \leq \Sigma + e_2^T L x(0)$ (condition (i) of

Theorem 2.7 is satisfied). Also, $L^2x(0) = [\,20 \;\; 8]^T \geq [\,10 \;\; 5\,]^T = m$ (condition (ii) of Theorem 2.7 is satisfied). Hence, m is (L,x(0)) reachable in 2 steps.

b) We proceed by solving the linear program set down in 3.4 for an optimal trajectory. Let $x(1) = x = [\,x_1 \;\; x_2\,]^T$.

maximize : $v^T(\,I - L^{-1})x = [1 \;\; -1]x$

subject to : i) $x(2) = m = [\,10\;5\,]^T$

$$\text{ii)}\quad 0 \leq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 8 \end{bmatrix}$$

$$0 \leq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

These contraints reduce to $10 \leq x_2 \leq 20$, and $5 \leq x_1 \leq 8$. The feasible region is therefore a rectangle in the $x_1 - x_2$ plane with vertices (5,10) , (5,20) , (8,10), and (8,20) . It is well known that the optimal solution is realized at one of these vertices [2]. Here is a table of values that the objective function takes on at each of the vertices of the rectagle:

Vertex	Objective
(5,10)	-5
(5,20)	-15
(8,10)	-2
(8,20)	-12

It is aparrent that the objective function is maximized at $x(1) = [8\ 10]^T$, and the corresponding yield by Result 3.3 equal to 21. We now use the fact that $x(t+1) = L(I - D_{h(t)})x(t)$ for $t = 0, 1$ and the optimal trajectory found above to obtain the optimal harvesting strategy –

$$\mathcal{S} = \{\ h'(0), h'(1)\ \} = \{ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{8} \\ 0 \end{bmatrix} \}.$$

Conclusion.

This paper has examined the problem of optimally contolling a population **P**, which is subject to harvesting and whose growth is approximated by a Leslie matrix, to a given *Reachable* size and distribution. The results exhibited will undoubtedly prove useful to a wildlife manager who makes use of the Leslie matrix model.

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Notation

$\Re$	the real numbers
$\mathcal{P}$	the positive integers
$\mathcal{N}$	the natural numbers
$M_{m,n}(\Re)$	*m* -by-*n* matices with entries from $\Re$
$M_n(\Re)$	*n* -by-*n* matrices with entries from $\Re$
$\rho(L)$	the spectral radius of $L \varepsilon M_n(\Re)$
I	identity matrix in $M_n(\Re)$
e_i	i^{th} standard basis vector in $M_{n,1}(\Re)$
A^T	transpose of $A \varepsilon M_{m,n}(\Re)$
A^{-1}	inverse of a nonsingular matrix $A \varepsilon M_n(\Re)$
$gcd(\mathcal{M})$	the greatest common divisor of $\mathcal{M}$, where $\mathcal{P} \supseteq \mathcal{M}$
x_i	the i^{th} component of $x \varepsilon M_{n,1}(\Re)$ (usually)